Envelop Equation for Bunched Beam

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December 26, 2006

Lawson introduced the beam envelop equation(*), though it assumes conditions that beam is almost continuous longitudinally, or energy of the beam is high enough and longitudinal space charge force is not necessary to be taken account. Thus, this envelop equation is not able to be used for short bunched beam like RFguns.

In this paper, we derive a beam envelop equation which are able to deal with low energy and short bunched beam. Note that we discuss only space charge force originated beam envelop trajectory.

1 Transverse Beam Envelop Equation

1.1 Electromagnetic Fields Produced by Pencil Beam

Electric fields, which are produced by an electron in linear uniform motion with velocity $\mathbf{v}$ are generally given as follows,

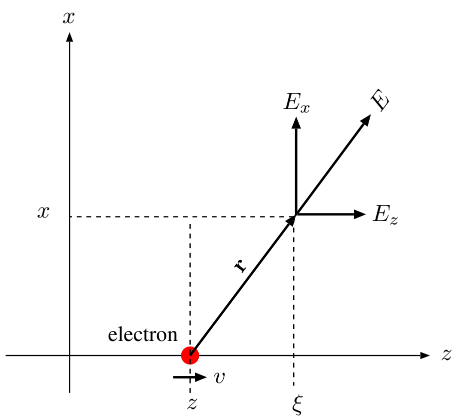

Figure 1:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0\gamma^2}\frac{-e\mathbf{r}}{\left[|\mathbf{r}|^2 - \frac{|\mathbf{v}\times\mathbf{r}|^2}{c^2}\right]^{3/2}}. \tag{1}$$

where, $\mathbf{r}$ is a vector from the present point of an electron, γ is relativistic factor of the electron.

Firstly, we start from Figure 1 in the rectangular coordinates, where the electron moves in uniform motion along the z-axis. Since,

$$\left[|\mathbf{r}|^2 - \frac{|\mathbf{v}\times\mathbf{r}|^2}{c^2}\right]^{3/2} = \left(\frac{1}{\gamma^2}x^2 + (\xi - z)^2\right)^{3/2}. \tag{2}$$

Then, E_x at $\mathbf{r}(\xi, x)$ which is produced by the electron is:

$$E_x = -\frac{e}{4\pi\epsilon_0\gamma^2}\frac{x}{\left(\frac{1}{\gamma^2}x^2 + (\xi - z)^2\right)^{3/2}}. \tag{3}$$

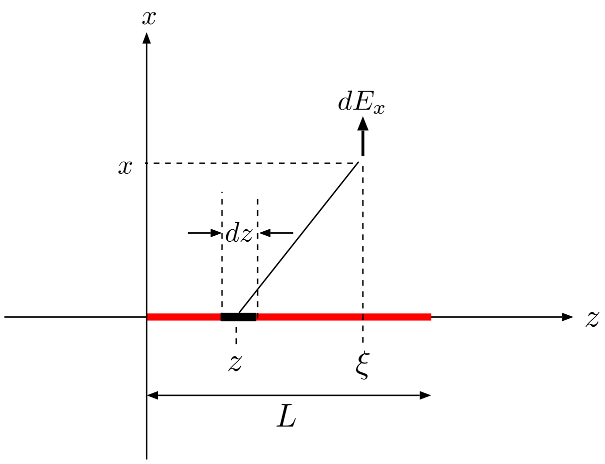

Figure 2:

Secondary, we assume as Figure 2 that electrons form one-dimensional pencil beam bunch along z-axis, a bunch length is L, and charge density is uniform longitudinally. E_x at $z = \xi$ can be calculated by integrating Equation (3) from tail to top of the bunch.

Here, dQ is charge in dz and total charge in the bunch is Q, then $dQ = \frac{Q}{L}dz$. Therefer, dE_x is:

$$dE_x = -\frac{1}{4\pi\epsilon_0\gamma^2}\cdot\frac{Q}{L}dz\cdot\frac{x}{\left(\frac{1}{\gamma^2}x^2 + (\xi - z)^2\right)^{3/2}} \tag{4}$$

Then, E_x is:

$$\begin{aligned} E_x &= \int_0^L dE_x = -\frac{Qx}{4\pi\epsilon_0\gamma^2 L}\int_0^L \frac{1}{\left(\frac{1}{\gamma^2}x^2 + (\xi - z)^2\right)^{3/2}}dz \\ &= \frac{-Q}{4\pi\epsilon_0 Lx}\left(\frac{L-\xi}{\sqrt{\frac{x^2}{\gamma^2} + (\xi - L)^2}} - \frac{-\xi}{\sqrt{\frac{x^2}{\gamma^2} + \xi^2}}\right) \end{aligned} \tag{5}$$

$E_{x\cdot center}$ at the center of the bunch can be obtained by replacing ξ with $L/2$ in the equation (5).

$$E_{x\cdot center} = \frac{-Q}{4\pi\epsilon_0 x}\frac{1}{\sqrt{\frac{L^2}{4} + \frac{x^2}{\gamma^2}}} \tag{6}$$

2 Beam Envelop Equation (transverse)

As shown in the Figure 3, we can obtain a transverse beam envelop equation by tracing the electron trajectory, which is initially located at double circle, x apart from the bunch center.

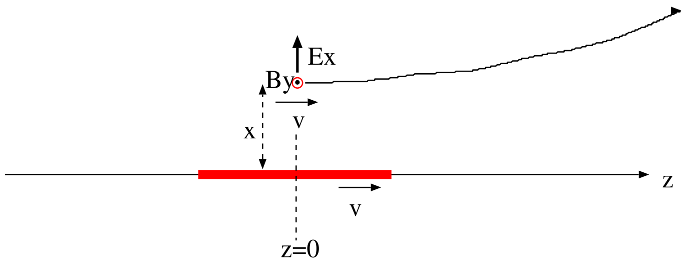

Figure 3:

An equation of motion of the electron is;

$$\frac{d\left(\gamma m_0 \mathbf{v}\right)}{dt} = -\frac{e}{m_0}\left(\mathbf{v}\times\mathbf{B} + \mathbf{E}\right) \tag{7}$$

From equation (7)

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{\gamma m_0}\left(\mathbf{v}\times\mathbf{B} + \mathbf{E} - \frac{(\mathbf{v}\cdot\mathbf{E})}{c^2}\mathbf{v}\right) \tag{8}$$

x component of equation (8) is;

$$\frac{dv_x}{dt} = -\frac{e}{\gamma m_0}\left(v_y B_z - v_z B_y + E_x - \frac{v_x E_x + v_y E_y + v_z E_z}{c^2}v_x\right) \tag{9}$$

We ignore v_y and define $v_z = \beta c$, then

$$\frac{dv_x}{dt} = -\frac{e}{\gamma m_0}\left(E_x - \beta c B_y - \left(\frac{v_x}{c}\right)^2 E_x - \frac{\beta v_x}{c} E_z\right) \tag{10}$$

Here, we discuss on effects of space charge force and acceleration by RF. Therefore, B_y is split into $B_{y \cdot charge}$ and $B_{y \cdot rf}$. We can ignore $\left(\frac{v_x}{c}\right)^2$. Then,

$$\frac{dv_x}{dt} = -\frac{e}{\gamma m_0}\left(E_{x \cdot charge} - \beta c B_{y \cdot charge} - \beta c B_{y \cdot rf} - \frac{\beta v_x}{c} E_{z \cdot rf}\right) \tag{11}$$

Using equation (6), space charge force acting on the electron is:

$$E_{x \cdot charge} = \frac{-Q}{4\pi\epsilon_0 x}\frac{1}{\sqrt{\frac{L^2}{4} + \frac{x^2}{\gamma^2}}} \tag{12}$$

$$B_{y \cdot charge} = \frac{v_z E_{x \cdot charge}}{c^2} = \frac{\beta}{c} E_{x \cdot charge} \tag{13}$$

Using $\frac{dx}{dt} = \beta c \frac{dx}{ds}$ and equation (12) (13), the transverse envelop equation is derived as follows;

$$\frac{d^2x}{ds^2} = \frac{eQ}{4\pi\epsilon_0 m_0 c^2 \gamma^3 \beta^2 x}\frac{1}{\sqrt{\frac{L^2}{4} + \frac{x^2}{\gamma^2}}} + \frac{e}{\gamma m_0 \beta c} B_{y \cdot rf} + \frac{e}{\gamma m_0 c^2} E_{z \cdot rf} \cdot \frac{dx}{ds} \tag{14}$$

In addition,when the bunch length L is long enough or energy γ is high enough, we can neglect $\frac{x^2}{\gamma^2}$ since $\frac{L^2}{4} \gg \frac{x^2}{\gamma^2}$. Then space charge term of this envelop equation become to coincide with those of the Lawson's envelop equation.

3 Longitudinal Beam Equation

Longitudinal beam envelop equation is also able to be obtained using a cylindrical beam instead of the pencil beam shown in Figure 3.

3.1 Electromagnetic Field Produced by Cylindrical Beam

At first, we assume a circular beam illustrated in Figure 4, whose line charge density of λ is uniform, and which moves along the z-axis with energy of γ. Electric field at point P, produced by a small section Δs which is a part of the circular beam is:

$$\Delta \mathbf{E} = \frac{1}{4\pi\epsilon_0\gamma^2}\frac{-\lambda \Delta s \mathbf{R'}}{\left[|\mathbf{R'}|^2 - \frac{|\mathbf{v}\times\mathbf{R'}|^2}{c^2}\right]^{3/2}} \tag{15}$$

Then, $E_{z \cdot ring}$ at point P is:

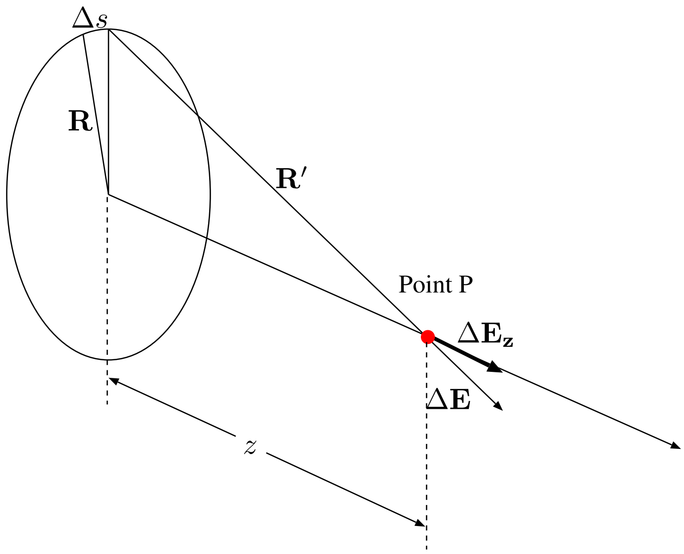

Figure 4:

$$E_{z \cdot ring} = \oint_s \Delta E_z = \frac{1}{2\epsilon_0 \gamma^2} \frac{-\lambda R z}{\left(z^2 + \frac{R^2}{\gamma^2}\right)^{3/2}} \tag{16}$$

Where R is radius of the circular beam.

Second, we replace the circular beam with a disk beam which area charge density is σ.

We divided the disk into concentric rings with width of Δr, radius of r, line charge density of $\lambda = \sigma \Delta r$. Using equation (16), $\Delta E_{z \cdot ring}$ at point P, produced by one concentric ring is:

$$\Delta E_{z \cdot ring} = \frac{1}{2\epsilon_0 \gamma^2} \frac{-\sigma \Delta r \cdot r z}{\left(z^2 + \frac{r^2}{\gamma^2}\right)^{3/2}} \tag{17}$$

Thus, $E_{z \cdot disk}$ is:

$$E_{z \cdot disk} = \int_0^R \frac{1}{2\epsilon_0 \gamma^2} \frac{-\sigma r z}{\left(z^2 + \frac{r^2}{\gamma^2}\right)^{3/2}} dr = \frac{\sigma z}{2\epsilon_0} \left(\frac{1}{\sqrt{z^2 + \frac{R^2}{\gamma^2}}} - \frac{1}{\sqrt{z^2}} \right) \tag{18}$$

Third, we consider a cylindrical beam instead of the disk beam illustrated in Figure 5, with beam bunch length of L, charge volume density of ρ. we divide the cylindrical beam into disk beams with longitudinal thickness of $\Delta\xi$, length between the disk and point P of ξ, charge area density of $\sigma = \rho \Delta\xi$. Point P

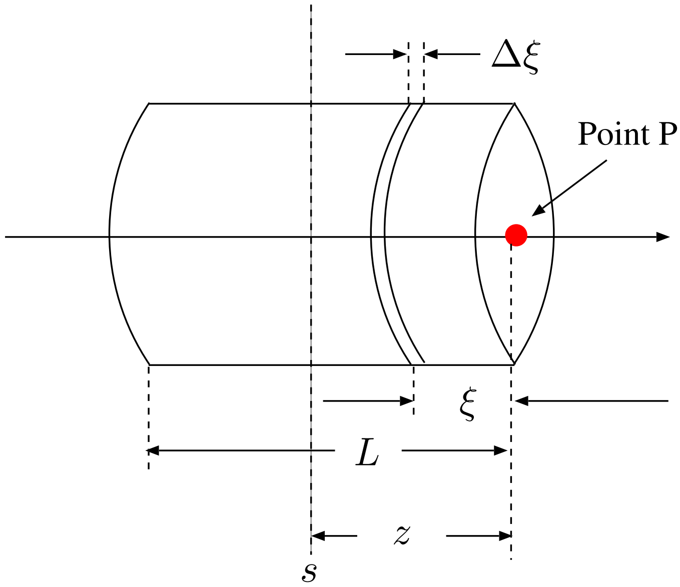

Figure 5:

is located at edge of the cylindrical beam, so $L = 2z$. $\Delta E_{z \cdot disk}$ at point P, produced by one disk is:

$$\Delta E_{z \cdot disk} = \frac{\rho}{2\epsilon_0} \left(\frac{\xi}{\sqrt{\xi^2 + \frac{R^2}{\gamma^2}}} - \frac{\xi}{\sqrt{\xi^2}} \right) \Delta\xi \tag{19}$$

Then, $E_{z \cdot column}$ is:

$$E_{z \cdot column} = \int_0^{2z} \frac{\rho}{2\epsilon_0} \left(\frac{\xi}{\sqrt{\xi^2 + \frac{R^2}{\gamma^2}}} - \frac{\xi}{\sqrt{\xi^2}} \right) \delta\xi \tag{20}$$

Using $Q = 2\rho\pi R^2 z$

$$E_{z \cdot column} = \frac{Q}{4\pi\epsilon_0 R^2 z} \left(\sqrt{4z^2 + \frac{R^2}{\gamma^2}} - \sqrt{\frac{R^2}{\gamma^2}} - 2z \right) \tag{21}$$

3.2 Beam Envelop Equation (longitudinal)

A longitudinal beam envelop equation is able to be derived in the same way of transverse equation.

An electron is located initially at point P as shown in the Figure 5. An equation of motion of the electron is;

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{\gamma m_0}\left(\mathbf{v}\times\mathbf{B} + \mathbf{E} - \frac{(\mathbf{v}\cdot\mathbf{E})}{c^2}\mathbf{v}\right) \tag{22}$$

z component of equation (22) is;

$$\frac{dv_z}{dt} = -\frac{e}{\gamma m_0}\left(v_x B_y - v_y B_x + E_z - \frac{v_x E_x + v_y E_y + v_z E_z}{c^2}v_z\right) \tag{23}$$

We ignore v_x, v_y, and define $v_z = v$ then;

$$\frac{dv}{dt} = -\frac{e}{\gamma^3 m_0}E_z \tag{24}$$

Here, v is sum of velocity of bunch center s and relative velocity between bunch center and the point P as shown in the Figure 5, then;

$$\frac{dv}{dt} = \frac{d^2 s}{dt^2} + \frac{d^2 z}{dt^2} = c\frac{d\beta}{dt} + \frac{d^2 z}{dt^2} \tag{25}$$

Now, we are deriving time evolution of relative position of point P to the bunch. Using $\frac{dz}{dt} = \beta c \frac{dz}{ds}$ and equation(21), longitudinal beam envelop equation is;

$$\begin{aligned}\frac{d^2 z}{ds^2} &= -\frac{e}{m_0 c^2 \gamma^3 \beta^2}E_{z\cdot charge} - \frac{e}{m_0 c^2\gamma^3\beta^2}E_{z\cdot rf} - \frac{1}{\beta}\frac{d\beta}{ds} \\ &= -\frac{eQ}{4\pi\epsilon_0 m_0 c^2\gamma^3\beta^2 R^2 z}\left(\sqrt{4z^2 + \frac{R^2}{\gamma^2}} - \sqrt{\frac{R^2}{\gamma^2}} - 2z\right) \\ &\quad - \frac{e}{m_0 c^2\gamma^3\beta^2}E_{z\cdot rf} - \frac{1}{\beta}\frac{d\beta}{ds}\end{aligned} \tag{26}$$

4 Simultaneous Beam Envelop Equations

We are able to derive simultaneous beam envelop equations from the transverse equation(14) and the longitudinal equation(26).

In the one-dimensional envelop equation, longitudinal beam bunch size L is constant in the transverse equation, and radial beam size R is constant in the longitudinal equation. Though in the simultaneous envelop equations, they are considered as variables. We apply following variation transformations;

$$\begin{cases} L = 2z & \text{for transverse equation.} \\ R = x & \text{for longitudinal equation.} \end{cases} \tag{27}$$

Then, simultaneous beam envelop equations are derived as follows;

$$
\begin{cases}
\frac{d^2x}{ds^2} = \frac{eQ}{4\pi\epsilon_0 m_0 c^2\gamma^3\beta^2 x}\frac{1}{\sqrt{z^2+\frac{x^2}{\gamma^2}}} + \frac{e}{\gamma m_0 \beta c}B_{y\cdot rf} + \frac{e}{\gamma m_0 c^2}E_{z\cdot rf}\cdot\frac{dx}{ds} \\
\dfrac{d^2z}{ds^2} = -\dfrac{eQ}{4\pi\epsilon_0 m_0 c^2\gamma^3\beta^2 x^2 z}\left(\sqrt{4z^2+\dfrac{x^2}{\gamma^2}} - \sqrt{\dfrac{x^2}{\gamma^2}} - 2z\right) \\
\qquad - \dfrac{e}{m_0 c^2\gamma^3\beta^2}E_{z\cdot rf} - \dfrac{1}{\beta}\dfrac{d\beta}{ds}
\end{cases}
\tag{28}
$$