Development of the Single-Bunch Instabilities Simulation Code SISR

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1 Introduction

The simulation code for single-bunch instabilities , SISR(Single-Bunc Instabilties in Storage Rings) was developped. SISR can treat not only localized impedance like cavities but also distributed broad-band impdance along rings such as small discontinuities of a beam pipe.

To incooperate distributed impedance, we use complex amplitude and an averaging method.

2 Basic Equations

The transverse position of i-th electron motion under the force F is represented with η_i as

$$\eta_i = \frac{x_i}{\sqrt{\beta}} \tag{1}$$

$$\theta = \frac{1}{\nu_0}\int^s \frac{ds'}{\beta}$$

$$\frac{d^2\eta_i}{d\theta^2} + (\nu_0 + \Delta\nu_i)^2\eta_i = \nu_0{}^2\,\beta^{\frac{3}{2}}\,\frac{F_i}{E} \quad . \tag{2}$$

Using η instead of (x,x'), we just need to know the values of β and α at each element and betatron phase difference between elemnts, which simlifys the discussion.

The force F_i comes from wake field is

$$F_i = e\sum_{j=1}^{N_p} q_j\, x_j\, \frac{d}{ds}W^{\perp}(z_j - z_i, s)$$

$$= e\,\beta^{\frac{1}{2}}\sum_{j=1}^{N_p} q_j\,\eta_j\,\frac{d}{ds}W^{\perp}(z_j - z_i, s) \quad . \tag{3}$$

$$\eta_i = \mathrm{Re}\left[a_i(\theta)\, e^{i\nu_0\theta}\right] = \frac{1}{2}\left(\, a_i(\theta)\, e^{i\nu_0\theta} + c.c\,\right) \tag{4}$$

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$$f_i(\theta) = e\,\beta^{\frac{1}{2}}\sum_{j=1}^{N_p} q_j\, a_j\, \frac{d}{ds}W^{\perp}(z_j - z_i, s) \tag{6}$$

Asumming that

$$\left|\frac{d^2 a_i}{d\theta^2}\right| << \left|2\, i\nu_0\frac{da_i}{d\theta}\right| \tag{7}$$

which means that the typical varying time of a_i is much smaller than the time λ_β/c, equation

$$\frac{da_i}{d\theta} = i\Delta\nu_i a_i + \frac{\nu_0}{2iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}}\, f_i\, \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)}$$

$$+ \frac{\nu_0}{2\, iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}}\, f_i\, e^{-2\, i\nu_0\theta'}\, \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)} \tag{8}$$

can be obtained.

The third term of the right hand side can be neglected because $\beta^{3/2}$ f does not have the Fourier component of the frequency $2\nu_0$.

Hence the equation

$$\frac{da_i}{d\theta} = i\Delta\nu_i a_i + \frac{\nu_0}{2\, iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}}\, f_i\, \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)} \tag{9}$$

is used to simulate distributed impdeance.

The integral in this equation is

$$\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}}\, f_i\, \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)} = \int_{s-\frac{\lambda}{2}}^{s+\frac{\lambda}{2}} \beta^{\frac{1}{2}}\, f_i\, \frac{ds'}{2\pi}$$

$$= \frac{1}{2\pi}\int_{s-\frac{C}{2}}^{s+\frac{C}{2}} \beta^{\frac{1}{2}}\, f_i\, ds' \frac{\lambda_\beta}{C}$$

From the approximation (7), we have

$$\int_{s-\frac{C}{2}}^{s+\frac{C}{2}} \beta^{\frac{1}{2}}\, f_i\, ds' = \sum_k \beta_k{}^{\frac{1}{2}}\int_{k\text{–}th\ element} f_i\, ds'$$

$$= e\sum_k \beta_k \sum_j q_j\, a_j\, W_k^{\perp}(z_j - z_i)$$

$$= e\sum_j q_j\, a_j \sum_k \beta_k\, W_k^{\perp}(z_j - z_i)$$

and, from the definition of λ_β,, $\frac{\lambda_\beta}{C} = \frac{1}{\nu_0}$.

The final form of the equation of transverse motion of particles are

$$\frac{da_i}{d\theta} = i\,\Delta\nu_i\, a_i + \frac{e}{4i\pi E_0}\sum_{j=1}^{N_p} q_j\, a_j \sum_{k=1}^{N_I} \beta_k\, W_k^{\perp}(z_j - z_i) \tag{10}$$

In the following, the symbol with superscript + is the value after and with - is the value before the passage of each element.

3 Difference Equations

3-1 Lattice with Distributed Broad-Band Impedance

The difference equations which used to simulate the electron motion in the lattice are

$$\Delta E_i^+ = \Delta E_i^- - U_0\,\frac{\Delta T}{T_0}\left(1 + \frac{\Delta E_i^-}{E_0}\right)^2 \tag{15}$$

$$z_i^+ = z_i^- - \alpha\,\frac{\Delta E_i^-}{E_0}\, c\,\Delta T \tag{16}$$

for longitudinal motion and

$$r_i^+ = r_i^- + \mathrm{Re}\left[\, f_i^-\, e^{-i\,\phi_{i,n}}\,\right]\Delta\theta \tag{12}$$

$$\phi_i^+ = \phi_i^- + \Delta\nu_i\,\Delta\theta + \frac{\mathrm{Im}\left[\, g_i^-\, e^{-i\phi_i^-} \right]}{\left\{ \dfrac{\left(r_i^+ + r_i^-\right)}{2} \right\}}\,\Delta\theta \qquad (13)$$

$$g_i^- = \frac{1}{4i\pi E_0} \sum_{j=1}^{N_p} q_j\, a_j^- \sum_{k=1}^{N_I} \beta_k\, W_k^{\perp}\left(z_j^- - z_i^-\right) \qquad (14)$$

for transverse motion, where $a = re^{i\phi}$ and $r = |a|$. and $\Delta\theta = 2\pi \dfrac{\Delta T}{T_0}$.

3-2 Localized Broad-Band Impedance

The effects of localized borad-band impdeance such as cavities are treated as

$$z_i^+ = z_i^- \qquad (17)$$

$$\Delta E_i^+ = \Delta E_i^- - e\sum_{j=1}^{N_p} q_j\, W^{\parallel}\left(z_j^- - z_i^-\right) \qquad (18)$$

$$a_i^+ = a_i^- - \frac{e}{iE_0}\, \beta^{\frac{1}{2}} \sum_{j=1}^{N_p} q_j\, x_j^-\, W^{\perp}\left(z_j^- - z_i^-\right) \qquad (19)$$

$$x = \beta^{\frac{1}{2}}\, \eta = \beta^{\frac{1}{2}}\, \mathrm{Re}\left[a\, e^{i\nu_0\theta} \right] \qquad (20)$$

3-3 Acceleration

Acceleration are teated as

$$z_i^+ = z_i^- \qquad (22)$$

$$\Delta E_i^+ = \Delta E_i^- + eV_a^- \qquad (23)$$

$$eV_a^- = eV_c \sin\left(2\pi f_{rf}\, \frac{z_i^-}{c} + \phi_c \right) \;. \qquad (24)$$

$$a_i^+ = a_i^- - i\, \frac{eV_a^-}{E_0}\, \mathrm{Im}\left[a_i^- e^{i\nu_0 u} \right] e^{-i\nu_0 u} \;. \qquad (25)$$

3-4 Radiation Excitation

Following equations are used for radiation excitation,

$$\Delta E_i^+ = \Delta E_i^- + \sqrt{\left(4\,\frac{\Delta T}{\tau_E}\right)} \left(\frac{\sigma_{E,0}}{E_0}\right) u_i \qquad (26)$$

$$a_i^+ = a_i^- + \sqrt{\left(4\,\frac{\Delta T}{\tau_\beta}\,\varepsilon_0\right)}\; v_i\, e^{i\,2\pi\, w_i} \;, \qquad (27)$$

where u, v is the Gaussian randum number and w is uniform randam number , ΔT is the time difference between each excitation.

4 Particle-In-Cell Method

Instaed of huge number of electrons, SISR uses superparticles. A Particle-In-Cell(PIC) method is used to make wake field and interact the particle with the wake field. The shape function used in SISR is

$$S(x) = \begin{cases} -\left(\dfrac{x}{\Delta x}\right)^2 + \dfrac{3}{4} & \left(|x| \le \dfrac{1}{2}\Delta x\right) \\ \dfrac{1}{2}\left|\dfrac{x}{\Delta x}\right|^2 - \dfrac{2}{3} & \left(\dfrac{1}{2}\Delta x \le |x| \le \dfrac{3}{2}\Delta x\right) \\ 0 & \left(\dfrac{3}{2}\Delta x \le |x|\right) \end{cases} \qquad (28)$$

and the distributions

$$\begin{Bmatrix} \rho(z_p) \\ a\rho(z_p) \\ x\rho(z_p) \end{Bmatrix} = \sum_{i=1}^{N} \begin{Bmatrix} q_i \\ a_i q_i \\ x_i q_i \end{Bmatrix} S\left(z_i - z_p\right) \qquad (29)$$

are evaluated on the mesh points whose position is represented with z_p,

And wake potentials appeared in above equations are evaluated at the mesh points using these distributions as

$$\begin{aligned} \frac{E_{BB}^{\perp}}{e}\left(z_p\right) &= -\sum_{j=1}^{N_p} q_j\, a_j \sum_{k=1}^{N_I} \beta_k W_k^{\perp}\left(z_j - z_p\right) \\ &= -\int a\rho(z')\sum_{k=1}^{N_I} \beta_k W_k^{\perp}\left(z' - z\right) dz' \end{aligned} \qquad (32)$$

$$\begin{aligned} \frac{E_{local}^{\perp}}{e}\left(z_p\right) &= -\sum_{j=1}^{N_p} q_j\, x_j^-\, W^{\perp}\left(z_j^- - z_i^-\right) \\ &= -\int x\rho(z')\, W^{\perp}\left(z_j^- - z_i^-\right) dz' \end{aligned} \qquad (33)$$

$$\begin{aligned} \frac{E^{\parallel}}{e}\left(z_p\right) &= -\sum_{j=1}^{N_p} q_j\, W^{\parallel}\left(z_j - z_p\right) \\ &= -\int \rho(z')\, W^{\parallel}\left(z' - z_p\right) dz' \;. \end{aligned} \qquad (34)$$

The potentials at the particles are obtained with

$$\frac{E}{e}\left(z_i\right) = \sum_{p=1}^{N_{mesh}} \frac{E}{e}\left(z_p\right) S\left(z_p - z_i\right) \qquad (35)$$

5 Evaluation of Wake Function

The three types of impedance models listed below are used to get wake funcions[1,2,3]

The three types of impedance models[3] listed below are used to get wake functions. For longitudinal,

$Z^{\parallel}$	$-i\,\omega\, Z_L^{\parallel}$	$Z_R^{\parallel}$	$\dfrac{Z_C^{\parallel}}{\sqrt{\omega}}(1+i)$
$W^{\parallel}$	$Z_L^{\parallel}\, c^2\, \dfrac{\partial\delta(z)}{\partial z}$	$Z_R^{\parallel}\, c\, \delta(z)$	$Z_C^{\parallel}\sqrt{\dfrac{2c}{\pi}}\, \lvert z\rvert^{-\frac{1}{2}}\, \theta(-z)$

and for transverse,

$Z^{\perp}$	$-i\, Z_L^{\perp}$	$\dfrac{Z_R^{\perp}}{\omega}$	$\dfrac{Z_C^{\perp}}{\omega\sqrt{\omega}}(1+i)$
$W^{\perp}$	$-Z_L^{\perp}\, c\, \delta(z)$	$-Z_R^{\perp}\, \theta(-z)$	$-Z_C^{\perp}\, 2\sqrt{\dfrac{2}{\pi c}}\, \lvert z\rvert^{\frac{1}{2}}\, \theta(-z)$

6 Conclusion

The single-bucnch instabilities simulation code SISR is developped. SISR is the particle-in-cell code and uses complex amplitude of electrons to describe electron motion. This method can treat distributed imipedance, such as small discontinuities. SISR also can treat localized impedance such as cavities.

References

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[3] S. Heifets, SLAC/AP-93(1992).